The McEliece Cryptosystem and Error-Correcting Codes

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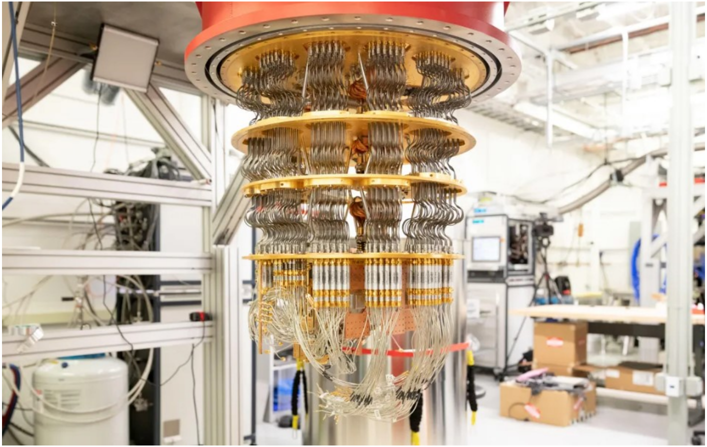
Background



- 1994: RSA was broken using Shor's Algorithm on a Quantum Computer.

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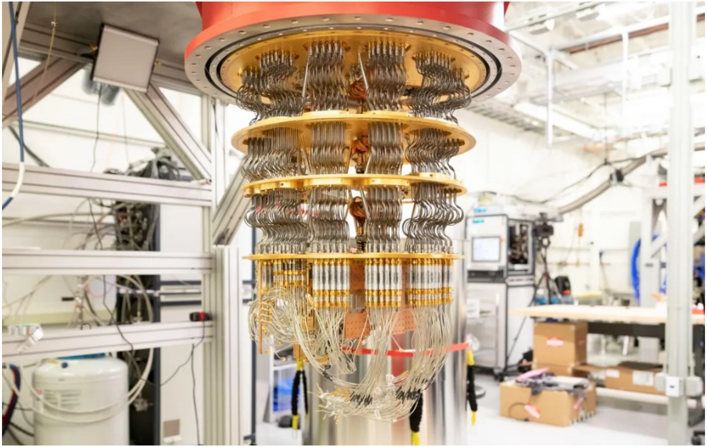

- 1994: RSA was broken using Shor's Algorithm on a Quantum Computer. But, don't worry $15 = 3 \times 5$.

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- July 2020: only four are left standing. One of these is the McEliece Cryptosystem, based on error-correcting codes.

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Goal:

The goal of this project is to understand the McEliece Cryptosystem, which utilizes Goppa Codes. Goppa Codes are a subclass of Linear Codes, and are related to Cyclic Codes and BCH Codes, which what we will focus on today.

Background on Error-Correcting Codes

The McEliece Cryptosystem based on Error-Correcting Codes

Alice	Public	Bob
	Eve	

The McEliece Cryptosystem based on Error-Correcting Codes

Alice	Public	Bob
		G
	Eve	

Bob: Chooses $k \times n$ generating matrix G,

The McEliece Cryptosystem based on Error-Correcting Codes

Alice	Public	Bob
		GP
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Bob: Chooses $k \times n$ generating matrix G, and $n \times n$ permutation matrix P,

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The McEliece Cryptosystem based on Error-Correcting Codes

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		4) Extract message $x_1 = (xS)$
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	Eve	

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	Eve	5) Compute $x_1 S^{-1} = (xS)S^{-1} = x$

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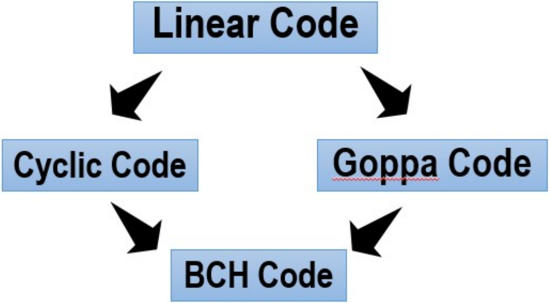

The McEliece Cryptosystem

The McEliece Cryptosystem uses a Goppa code of length 1024 that can correct 50 errors. In this case, Eve has $\binom{1024}{50} \approx 3x10^{85}$ possible locations of the errors.

Linear Code

- **Definition:** An $[n, k, d]$ linear code is a vector space with dimension k and length n over a field $\mathbb{F}$, where the combination of any two codewords is always a codeword, with minimum distance d between two codewords.

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- Given a linear code C, t errors corrected if $d(C) \geq 2t + 1$

Linear Code

- **Definition:** An $[n, k, d]$ linear code is a vector space with dimension k and length n over a field $\mathbb{F}$, where the combination of any two codewords is always a codeword, with minimum distance d between two codewords.
- **Parity Check Matrix (PCM Hamming Code):** Given generating matrix $G = [I_k, P] \in C$, then $H = [-P^T, I_{n-k}]$ is a parity check matrix for C if and only if $rH^T = 0$
- **Coset:** Given linear code C, and n-dimensional vector r, a coset is the set of $r + C$. The vector with minimum Hamming weight is the **coset leader**.
- Given a linear code C, s errors detected if minimum distance $d(C) \geq s + 1$
- Given a linear code C, t errors corrected if $d(C) \geq 2t + 1$
- **Syndrome:** This is defined as $S(r) = rH^T$

Linear Hamming Code

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  4. message $= r -$ coset leader.

Linear Hamming Code

Example: Consider a $[4, 2, 2]$ linear code with G .

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generating matrix $G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$,

Here is the lookup table for decoding received words.

$$\begin{array}{l} (0,0,0,0)(1,0,0,1)(0,1,0,1)(1,1,0,0) \\ (1,0,0,0)(0,0,0,1)(1,1,0,1)(0,1,0,0) \\ (0,0,1,0)(1,0,1,1)(0,1,1,1)(1,1,1,0) \\ (0,0,1,1)(1,0,1,0)(0,1,1,0)(1,1,1,1) \end{array}$$

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Alice encodes message $x = [1, 1]$ by computing $xG = [1, 1, 0, 0]$. She sends to Bob through a noisy channel, and Bob receives $r = (1, 1, 1, 0)$.
Next, Bob must **DECODE**, by calculating the syndrome $S(r) = rH^T$.

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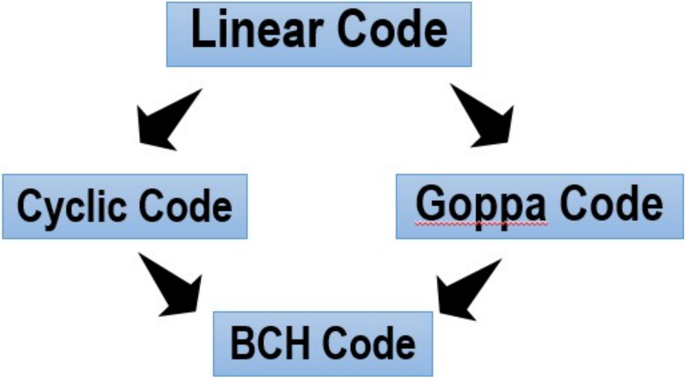
Cyclic Code



Cyclic Code

A code is said to be a **cyclic code** if it contains the property

$$(c_1, c_2, ..., c_{n-1}, c_n) \in C \Longleftrightarrow (c_n, c_1, c_2, \ldots, c_{n-1}) \in C$$

Given $g(X) = g_0 + g_1 X + \cdots + g_{n-1} X^{n-1} + g_{n-k} X^{n-k}$, and $h(X) = h_0 + h_1 X + \cdots + h_{k-1} X + h_k X^k$ (where $g(X)h(X) = X^n - 1$), we formulate $k \times n$ generating matrix G

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Cyclic Code

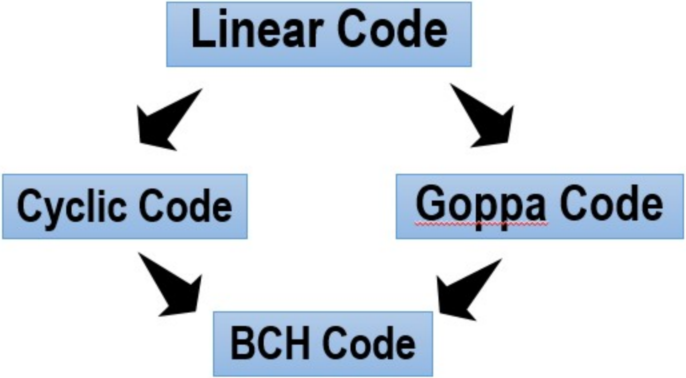
Example: Here is an $[n, k, d] = [7, 3, 4]$ cyclic code C.

$$X^7 - 1 = g(X)h(X) = \underbrace{(X^4 + X^2 + X + 1)}_{g(X)}\underbrace{(X^3 + X + 1)}_{h(X)}.$$

Then 3×7 generating matrix G and 4×7 parity check matrix H are:

$$G = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

BCH (Bose-Chaudhuri-Hocquenhem) Code



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Theorem

Let C be an $[n, k, d]$ cyclic code over $\mathbb{F}_{q=p^m}$, where $p \nmid n$. Let α be a primitive n-th root of unity, and let $g(X)$ be a generating polynomial for C. Suppose there exist integers ℓ and δ such that

$$g(\alpha^{\ell}) = g(\alpha^{\ell+1}) = \cdots = g(\alpha^{\ell+\delta}) = 0$$

Then the minimum distance $d \geq \delta + 2$.

The parity check matrix H is

$$H = \begin{bmatrix} 1 & \alpha^{k+1} & \alpha^{2(k+1)} & \dots & \alpha^{(n-1)(k+1)} \\ 1 & \alpha^{k+2} & \alpha^{2(k+2)} & \dots & \alpha^{(n-1)(k+2)} \end{bmatrix}$$

BCH (Bose-Chaudhuri-Hocquenhem) Code

How to DECODE a received word r for one error?

1. Calculate $rH^T = (s_1, s_2)$.
2. If $s_1 = 0$, then no error (r is a codeword).
3. If $s_1 \neq 0$, compute $\frac{s_2}{s_1} = \alpha^{j-1}$, where j is position of error.
4. $r - e =$ codeword

BCH (Bose-Chaudhuri-Hocquenhem) Code

Example: Consider a $[7, 1, 7]$ BCH code with generating polynomial $g(X) = X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$. There are two codewords: $(0, 0, 0, 0, 0, 0, 0)$ and $(1, 1, 1, 1, 1, 1, 1)$. Suppose Bob receives $r = (1, 1, 1, 0, 1, 1, 1)$. Detect and correct the error!

Solution: Since $rH^T = (s_1, s_2)$, we see

$$rH^T = (1, 1, 1, 0, 1, 1, 1) \begin{bmatrix} 1 & 1 \\ \alpha & \alpha^2 \\ \alpha^2 & \alpha^4 \\ \vdots & \vdots \\ \alpha^6 & \alpha^{12} \end{bmatrix} = (s_1, s_2) = \left(\alpha^3, \alpha^6\right)$$

Since $s_1 \neq 0$, we calculate $\frac{s_2}{s_1} = \frac{\alpha^6}{\alpha^3} = \alpha^3$. Therefore, $j - 1 = 3$, so the error position is at $j = 4$. Finally, we see

$$r - e = \underbrace{(1, 1, 1, 0, 1, 1, 1)}_{\text{received word}} - \underbrace{(0, 0, 0, 1, 0, 0, 0)}_{\text{error vector}} = \underbrace{(\underbrace{1}_{\text{message}}, 1, 1, 1, 1, 1, 1)}_{\text{codeword}}$$

References

- *Introduction to Cryptography with Coding Theory*, by W. Trappe, L. Washington, Pearson, 2nd edition, 2006
- *Fundamentals of Error-Correcting Codes*, by V. Pless, W.C. Huffman, Cambridge University Press, 2003

Conclusion